(d) From $|a_{1\to 2}(t)|^2$ to Rate of Transition $\lambda_{1\to 2}$

Eq.(17):

$$|a_{1\to 2}(t)|^2 = \mathcal{E}_0^2 e^2 |z_{21}|^2 \frac{\sin^2\left[\frac{(E_2 - E_1 - \hbar\omega)t}{2\hbar}\right]}{(E_2 - E_1 - \hbar\omega)^2}$$

single ω (monochromatic)

$$= \underbrace{\left(\frac{1}{2}\varepsilon_0 \mathcal{E}_0^2\right)} \frac{2}{\varepsilon_0} e^2 |z_{21}|^2 \left(\begin{array}{c}\text{shapely peaked at } \hbar\omega = E_2 - E_1 \\ \text{\& area under curve grows} \sim t\end{array}\right) \quad (20)$$

$$= U_\omega \cdot \frac{2}{\varepsilon_0} e^2 |z_{21}|^2 \underbrace{\left(\quad \cdots \quad \right)}_{\text{picks up light at the right } \hbar\omega\ (= E_2 - E_1)}$$

U_ω: energy density [energy per unit volume] ($\sim$ intensity of incident light) monochromatic of angular frequency ω

ε_0: ε_0 in "$4\pi\varepsilon_0$"

Generalize to Non-monochromatic incident light: a more practical situation

- Incident light has a spread in frequency

$U(\omega)\,d\omega$ = energy density of incident light in angular freq range $\omega \to \omega + d\omega$

(i.e. use a continuum description)

Idea: $\omega_1, \omega_2, \cdots, \omega'$ in the range

$$\underbrace{\begin{array}{ccc} \omega_1 & & \omega' \\ \updownarrow & & \updownarrow \\ |a_2(t)|^2_{\omega_1} & & |a_2(t)|^2_{\omega'} \end{array}}$$

$\underbrace{\text{add}^{\dagger} \text{ them up}}_{\text{involving}}$ to get $|a_2(t)|^2$

involving $\displaystyle\int_{\text{range of } \omega \text{ in incident light}} (\cdots)\, U(\omega)\, d\omega$

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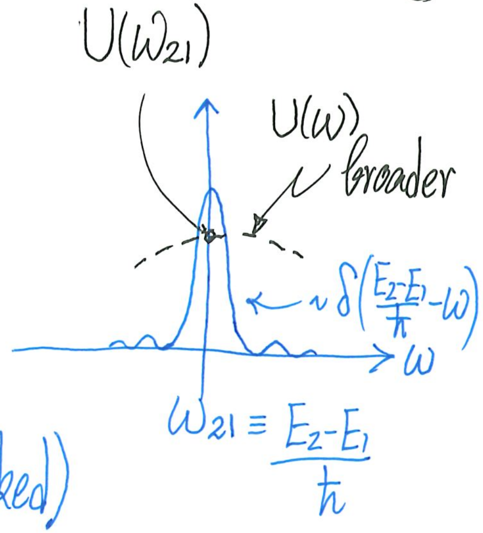
† This is referred to as "incoherent perturbation" that works for non-monochromatic incoherent EM waves

$$\underbrace{|a_2(t)|^2_{\omega\to\omega+d\omega}}_{\text{contributions from } \omega\to\omega+d\omega} = \frac{2}{\varepsilon_0} e^2 |z_{21}|^2 \underbrace{\frac{\sin^2\left[\frac{(E_2-E_1-\hbar\omega)}{2\hbar}t\right]}{(E_2-E_1-\hbar\omega)^2}}_{\frac{\pi}{2\hbar^2} t\,\delta\left(\frac{E_2-E_1}{\hbar}-\omega\right)} \cdot U(\omega)\,d\omega$$

(sharply peaked)



$$|a_2(t)|^2 = \frac{2e^2}{\varepsilon_0} |z_{21}|^2 \int \frac{\sin^2\left[\frac{(E_2-E_1-\hbar\omega)}{2\hbar}t\right]}{(E_2-E_1-\hbar\omega)^2} \cdot U(\omega)\,d\omega$$

$$= \frac{2e^2}{\varepsilon_0} |z_{21}|^2 \frac{\pi}{2\hbar^2} t \cdot U(\omega_{21})$$

where $\omega_{21} \equiv \frac{E_2-E_1}{\hbar}$ (the atom's property)

$$= \frac{\pi}{\varepsilon_0 \hbar^2} \underbrace{U(\omega_{21})}_{\text{picks up strength (intensity) of light at } \omega_{21} \text{ in the beam}} \underbrace{e^2 |z_{21}|^2}_{\text{electric dipole moment matrix element (selection rule)}} \cdot t \qquad (21)$$

← linear in time (because used area under curve) (instead of value at the peak)

at ω_{21}: at the correct frequency

Make physical sense

involve atomic states and z (or x & y) of electron in the middle

$$|a_{1\to 2}(t)|^2 = \underbrace{\frac{2e^2}{\varepsilon_0}\frac{\pi}{2\hbar^2}}_{\text{constants (from QM derivation)}} \cdot \underbrace{|z_{21}|^2}_{\text{selection rule}} \cdot \underbrace{U(\omega_{21})}_{\text{proportional to how strong the incident light is at the correct frequency}} \cdot t \qquad (21)$$

Absorption

constants (from QM derivation)

selection rule

proportional to how strong the incident light is at the correct frequency

t: proportional to time$^+$ t that light interacts with atom

$^+$ time t is often limited by factors such as collision with other atoms, thus don't be carried away that $|a_{1\to 2}(t)|^2$ can keep on increasing. Often, $|a_{1\to 2}(t)|^2$ is small, as we did perturbation.

$$\frac{|a_{1\to2}(t)|^2}{t}$$ is a quantity of unit $1/\text{time}$ (almost equal to transition rate)

Applying the same argument to stimulated emission gives

$$\boxed{|a_{2\to1}(t)|^2 = \frac{2e^2}{\varepsilon_0}\frac{\pi}{2\hbar^2}\cdot|z_{21}|^2\cdot U(\omega_{21})\cdot t} \qquad (22)$$

Same expression as $|a_{1\to2}(t)|^2$, as same $|z_{21}|^2$ enters

- The Rate at which transition $(1 \to 2)$ occurs

  $\equiv$ Transition Probability per unit time

  $$\equiv \lambda_{1\to 2} = \frac{|a_2(t)|^2}{t} = \frac{\pi e^2}{\varepsilon_0 \hbar^2} U(\omega_{21}) \, |z_{21}|^2 \quad (23)$$

  (non-monochromatic) (for $\hat{z}$-polarized light)

  (1: lower; 2: higher)

  averaging over polarizations and propagation directions†

Generally, Transition rate (absorption)

$$\boxed{\lambda_{1\to 2} = \frac{\pi e^2}{3\varepsilon_0 \hbar^2} U(\omega_{21}) \, |r_{21}|^2} \quad (24)$$

- $|r_{21}|^2 \equiv |\vec{r}_{21}|^2$ with $\vec{r}_{21} = \int \psi_2^*(\vec{r}) \, \vec{r} \, \psi_1(\vec{r}) \, d^3r$ ← a vector (generally complex)

  $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$

- $e^2 |\vec{r}_{21}|^2 \equiv |\vec{\mu}_{21}|^2$ (reminds us that it is electric dipole moment that matters)

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† Don't worry about the details from (23) to (24). They carry the same physics.

Following same argument,

Transition rate (stimulated emission)

$$\boxed{\lambda_{2\to1} = \frac{\pi e^2}{3\varepsilon_0\hbar^2}\, U(\omega_{21})\, |r_{21}|^2} \qquad (25)$$

$\lambda_{\substack{1\to2\\2\to1}} \propto U(\omega_{21})\,|r_{21}|^2$

"Applied QM common sense"

Between two states (one upper & one lower), $|r_{21}|^2$ in (24) & (25) is the same

$\therefore$ Between two states under the same condition [$U(\omega_{21})$ the same]

$$\boxed{\begin{array}{l}\text{Transition rate of Stimulated Absorption} \\ \quad = \text{Transition rate of stimulated emission}\end{array}} \qquad (26)$$

for an atom

- A result of QM (Applied QM common sense)
- QM gives absorption and stimulated emission

good

Given (26), if there are only 2 states (one lower & one higher), will it be possible to use light to excite to a situation where there are more atoms in state 2 than in state 1 in a gas of atoms?

Keywords: population inversion, 3-level & 4-level lasers

Another Applied QM common sense

Hint: Start with $\frac{dN_1}{dt}$ and $\frac{dN_2}{dt}$, $N_1 + N_2 = N$ = total # atoms

Summary and Remarks

- Absorption and Stimulated Emission can be readily explained by Schrödinger QM
- What about Spontaneous Emission (no applied $U(\omega_{21})$) ?
- Eqs. (23), (24), (25) are special forms of $\underbrace{\text{the Fermi Golden Rule}}_{\text{gives transition rates (derived by Dirac)}}$
- How about from initial state to a group of final states (~ degeneracy at E_2)

  ← many "states 2" at $E_2 - E_1 = \hbar\omega$ (typical situation in a solid)

  1 —

  Same idea: Sum up contributions of $\underbrace{\text{many final states}}$

  $D(E)dE$ = # states with energy in interval $E \to E + dE$

- $\underbrace{\lambda_{1\to 2}}$ generalized to $\underbrace{\lambda_{\text{initial}\to(\text{many final states})}}_{\text{related to spectroscopic absorption coefficient}}$

related to spectroscopic $\underbrace{\text{absorption coefficient}}_{\text{a measurable quantity}}$

- $\dfrac{1}{\lambda}$ is $\underbrace{\text{a time}}_{\text{characteristic of an atomic state}}$ (related to life time)

This is real Applied Quantum Mechanics!